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On a Certain Completely Integrable System of Linear Partial Differential Equations.

By E. J. Wilczynski.

Introduction.

From the point of view of projective differential geometry, developables and ruled surfaces require a special discussion, on account of their many exceptional properties. These exceptional properties manifest themselves analytically by the fact that, for these surfaces, the differential invariants of lowest order are equal to zero. If a surface is neither a developable nor a ruled surface, its absolute invariants of lowest order are those which have been denoted by the author by I and J. In a certain sense, then, those surfaces for which I and J are identically equal to zero, are to be regarded as constituting an especially simple and fundamental class.

I have shown recently,[*] that these surfaces are the integral surfaces of the completely integrable system of partial differential equations

$$(S) \quad \begin{cases} \dfrac{\partial^2 y}{\partial u^2} + 2\dfrac{\partial y}{\partial v} + (c_0 u + c_1)\, y = 0, \\ \dfrac{\partial^2 y}{\partial v^2} + 2\dfrac{\partial y}{\partial u} + (c_0 v + c_2)\, y = 0, \end{cases}$$

where c_0, c_1 and c_2 are constants, and I have studied in detail the case $c_0 = 0$. The system (S) is then capable of integration by means of elementary functions, and the corresponding surfaces, for which, of course, $I = J = 0$, have the further property of being invariant under a continuous group of projective transformations. If $c_0 \neq 0$, no such group exists, which leaves the surface invariant, and the problem of integrating (S) can not be solved in such a simple manner.

The directrix curves[†] of any integral surface of (S) form a conjugate net, whether c_0 vanishes or not. Upon this remark may be based a transformation theory of system (S) which we hope to discuss on some future occasion. The present paper, however, is merely devoted to the integration of system (S) in the case $c_0 \neq 0$.

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[*] "On a Certain Class of Self-projective Surfaces," *Transactions of the American Mathematical Society*, October, 1913.

[†] For the notion of directrix curves, cf. a paper of mine in the *Transactions of the American Mathematical Society*, Vol. IX (1908), pp. 114–120.

In order to accomplish this integration, we find it necessary to solve the following problem: To integrate a certain completely integrable system of non-homogeneous linear partial differential equations, when the solutions of the corresponding homogeneous system are known. Since I have been unable to find any discussion of this question in the literature, the first two sections of this paper are devoted to a solution of this problem in so far as it is needed for what follows. It then becomes possible to prove the existence of various kinds of solutions of system (S). The most interesting of these can be written as an exponential function, multiplied by a series of positive integral powers of c_0, the coefficients of this power-series being *polynomials* in u and v. A number of methods will be given for calculating these polynomials and for studying some of their properties, but I have been able to find a perfectly explicit formula for only some of them.

The solutions of (S) are found to satisfy integral equations of a peculiar kind, involving integrals of exact differentials. A more detailed discussion of such integral equations will be given in a separate paper.

§ 1. *Integrability Conditions for a Non-Homogeneous System of Linear Partial Differential Equations of the Second Order.*

Let us consider the non-homogeneous system of partial differential equations

$$(L) \quad \begin{cases} y_{uu} + 2a\,y_u + 2b\,y_v + c\,y + d = 0, \\ y_{vv} + 2a'y_u + 2b'y_v + c'y + d' = 0, \end{cases}$$

where

$$y_u = \frac{\partial y}{\partial u}, \quad y_v = \frac{\partial y}{\partial v}, \quad y_{uu} = \frac{\partial^2 y}{\partial u^2}, \quad y_{vv} = \frac{\partial^2 y}{\partial v^2},$$

and where $a, b, \ldots, d'$ are analytic functions of u and v. By differentiating these equations, we obtain, for the third-order derivatives of y, the following unique expressions:

$$\begin{aligned} y_{uuu} &= p_1 y_{uv} + p_2 y_u + p_3 y_v + p_4 y + p_5, \\ y_{uuv} &= q_1 y_{uv} + q_2 y_u + q_3 y_v + q_4 y + q_5, \\ y_{uvv} &= r_1 y_{uv} + r_2 y_u + r_3 y_v + r_4 y + r_5, \\ y_{vvv} &= s_1 y_{uv} + s_2 y_u + s_3 y_v + s_4 y + s_5, \end{aligned}$$

where

$$q_1 = -2a, \; q_2 = 4a'b - 2a_v, \; q_3 = 4bb' - 2b_v - c, \; q_4 = 2bc' - c_v, \; q_5 = 2bd' - d_v,$$
$$r_1 = -2b', \; r_2 = 4aa' - 2a'_u - c', \; r_3 = 4a'b - 2b'_u, \; r_4 = 2a'c - c'_u, \; r_5 = 2a'd - d'_u,$$

and where the values of p_i and s_i are immaterial for our present purpose.

If we form the fourth-order derivatives of y, the expressions for y_{uuuu} and y_{vvvv} are unique, and the two expressions found for y_{uuuv}, viz., $\frac{\partial y_{uuu}}{\partial v}$ and $\frac{\partial y_{uuv}}{\partial u}$, will be identical without imposing any further condition, since each of them is the result of differentiating the first equation of (L) once with respect to u and once with respect to v. Similarly, the equation

$$\frac{\partial y_{uvv}}{\partial v} = \frac{\partial y_{vvv}}{\partial u}$$

will be satisfied without imposing any conditions on the coefficients of (L). The equality of the two expressions $\frac{\partial y_{uuv}}{\partial v}$ and $\frac{\partial y_{uvv}}{\partial u}$, however, gives rise to such conditions. In fact we find

$$\begin{aligned}\frac{\partial y_{uuv}}{\partial v} = & \left(\frac{\partial q_1}{\partial v} + q_1 r_1 + q_2\right) y_{uv} + \left(\frac{\partial q_2}{\partial v} + q_1 r_2 - 2 a' q_3\right) y_u \\ & + \left(\frac{\partial q_3}{\partial v} + q_1 r_3 - 2 b' q_3 + q_4\right) y_v + \left(\frac{\partial q_4}{\partial v} + q_1 r_4 - c' q_3\right) y + \frac{\partial q_5}{\partial v} + q_1 r_5 - d' q_3, \\ \frac{\partial y_{uvv}}{\partial u} = & \left(\frac{\partial r_1}{\partial u} + r_1 q_1 + r_3\right) y_{uv} + \left(\frac{\partial r_2}{\partial u} + r_1 q_2 - 2 a r_2 + r_4\right) y_u \\ & + \left(\frac{\partial r_3}{\partial u} + r_1 q_3 - 2 b r_2\right) y_v + \left(\frac{\partial r_4}{\partial u} + r_1 q_4 - c r_2\right) y + \frac{\partial r_5}{\partial u} + r_1 q_5 - d r_2\end{aligned}$$

In order that system (L) may have four linearly independent solutions, the corresponding coefficients of y_{uv}, y_u, y_v, y and y^0 in these two expressions must be equal.* *We obtain in this way the following five integrability conditions:*

$$(I) \begin{cases} a_v - b'_u = 0, \\ a'_{uu} + c'_u - 2 a' a_u - 2 a a'_u - (a_{vv} + 2 b' a_v - 2 b a'_v - 4 a' b_v) = 0, \\ b'_{uu} + 2 a b'_u - 2 a' b_u - 4 b a'_u - (b_{vv} + c_v - 2 b b'_v - 2 b' b_v) = 0, \\ c'_{uu} - 4 c a'_u - 2 a' c_u + 2 a c'_u - (c_{vv} - 4 c' b_v - 2 b c'_v + 2 b' c_v) = 0, \\ d'_{uu} + 2 a d'_u - 2 a' d_u - 4 a'_u d - c' d - (d_{vv} + 2 b' d_v - 2 b d'_v - 4 b_v d' - c d') = 0. \end{cases}$$

Thus, the integrability conditions for the non-homogeneous system (L) differ from those for the corresponding homogeneous system only by the presence of the last of these five conditions.†

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* Otherwise y would also satisfy an equation of the form

$$y_{uv} + 2 a'' y_u + 2 b'' y_v + c'' y + d'' = 0.$$

The system obtained by adding this equation to (L) has at most three linearly independent solutions.

† E. J. Wilczynski, "Projective Differential Geometry of Curved Surfaces" (First Memoir), *Transactions of the American Mathematical Society*, Vol. VIII (1907), p. 245.

§ 2. *Integration by Quadratures of the Non-Homogeneous System in Terms of the Solutions of the Corresponding Homogeneous System.*

If the conditions (I) are satisfied, (L) is a completely integrable system, and the corresponding homogeneous system

$$(H) \quad \begin{cases} y_{uu} + 2\,a\,y_u + 2\,b\,y_v + c\,y = 0, \\ y_{vv} + 2\,a' y_u + 2\,b' y_v + c' y = 0 \end{cases}$$

will also be completely integrable. Therefore, (H) will possess precisely four linearly independent solutions. Let $y^{(1)}$, $y^{(2)}$, $y^{(3)}$, $y^{(4)}$ be four such linearly independent solutions of (H), so that

$$c_1 y^{(1)} + c_2 y^{(2)} + c_3 y^{(3)} + c_4 y^{(4)}$$

will be its general solution if $c_1, \ldots, c_4$ are arbitrary constants. Let y be the general, and Y any particular solution of (L) Then $y - Y$ will be a solution of (H), so that

$$y = Y + c_1 y^{(1)} + c_2 y^{(2)} + c_3 y^{(3)} + c_4 y^{(4)}$$

Consequently, if we assume the solutions of (H) as known, it suffices to find any particular solution of (L) in order to integrate (L) completely.

We use the method of variation of constants to find such a particular solution of (L). Let us write Y in the form

$$Y = \sum_{k=1}^{4} l^{(k)} y^{(k)}, \tag{1}$$

where $l^{(1)}, \ldots, l^{(4)}$ are functions, yet to be determined, of u and v. We shall have

$$Y_u = \Sigma\, l^{(k)} y_u^{(k)}, \qquad Y_v = \Sigma\, l^{(k)} y_v^{(k)},$$

if we impose upon $l^{(1)}, \ldots, l^{(4)}$ the conditions

$$\Sigma\, l_u^{(k)} y^{(k)} = \Sigma\, l_v^{(k)} y^{(k)} = 0,$$

or

$$\Sigma\, y^{(k)}\, d\, l^{(k)} = 0. \tag{2}$$

We then find

$$Y_{uu} = \Sigma\,(l^{(k)} y_{uu}^{(k)} + l_u^{(k)} y_u^{(k)}), \qquad Y_{vv} = \Sigma\,(l^{(k)} y_{vv}^{(k)} + l_v^{(k)} y_v^{(k)}),$$

so that Y will be a solution of (L), if, and only if,

$$\Sigma\, l_u^{(k)} y_u^{(k)} = -\,d, \qquad \Sigma\, l_v^{(k)} y_v^{(k)} = -\,d'. \tag{3}$$

We may write

$$Y_{uv} = \Sigma\, l^{(k)} y_{uv}^{(k)},$$

if we impose the further conditions

$$\Sigma\, l_v^{(k)} y_u^{(k)} = 0, \qquad \Sigma\, l_u^{(k)} y_v^{(k)} = 0. \tag{4}$$

But equations (3) and (4) are equivalent to the following:

$$\Sigma y_u^{(k)}\, d\, l^{(k)} = -d \cdot du, \quad \Sigma y_v^{(k)}\, d\, l^{(k)} = -d' \cdot dv, \tag{5}$$

so that we have in (2) and (5) a system of three equations for the four differentials $d\, l^{(k)}$. We add to these equations a fourth, viz.,

$$\Sigma y_{uv}^{(k)}\, d\, l^{(k)} = -(A\, du + B\, dv)$$

where we shall determine the (as yet unknown) functions A and B of u and v in such a way that the values of $d\, l^{(k)}$ obtained from these four equations shall be exact differentials.

Let us write these equations as follows:

$$\left.\begin{aligned} \Sigma y_{uv}^{(k)}\, d\, l^{(k)} &= -(A\, du + B\, dv), \\ \Sigma y_u^{(k)}\, d\, l^{(k)} &= -d \cdot du, \\ \Sigma y_v^{(k)}\, d\, l^{(k)} &= -d' \cdot dv, \\ \Sigma y^{(k)}\, d\, l^{(k)} &= 0, \end{aligned}\right\} \tag{6}$$

and denote the determinant of the left members by Δ, so that

$$\Delta = |\, y_{uv},\ y_u,\ y_v,\ y\,|$$

Then Δ is not identically equal to zero. For, if it were, (H) would have at most three linearly independent solutions, and $y^{(1)}, \ldots, y^{(4)}$ could not be linearly independent.

We may write

$$\Delta = \Sigma y_{uv}^{(k)}\, \lambda^{(k)} = \Sigma y_u^{(k)}\, \mu^{(k)} = \Sigma y_v^{(k)}\, \nu^{(k)} = \Sigma y^{(k)}\, \rho^{(k)},$$

where $\lambda^{(k)}$, $\mu^{(k)}$, etc., are the third-order minors of Δ, and may be written as follows:

$$\begin{aligned} \lambda^{(1)} &= +\,|y_u,\ y_v,\ y|, & \mu^{(1)} &= -\,|y_{uv},\ y_v,\ y|, \\ \nu^{(1)} &= +\,|y_{uv},\ y_u,\ y|, & \rho^{(1)} &= -\,|y_{uv},\ y_u,\ y_v|, \end{aligned}$$

etc., the notation adopted being self-explanatory.

The solution of (6) for $dl^{(k)}$ gives

$$\Delta\, d\, l^{(k)} = -\lambda^{(k)}\,(A\, du + B\, dv) - \mu^{(k)}\, d\ du - \nu^{(k)}\, d'\ dv$$

But we find

$$\begin{aligned} \lambda_u^{(1)} &= |\, y_{uu},\ y_v,\ y\,| + |\, y_u,\ y_{uv},\ y\,| + |\, y_u,\ y_v,\ y_u\,| \\ &= |-2a y_u - 2b y_v - c y,\ y_v,\ y\,| + |\, y_u,\ y_{uv},\ y\,| = -2a\lambda^{(1)} - \nu^{(1)}, \end{aligned}$$

and similarly,

$$\lambda_u^{(k)} = -2a\lambda^{(k)} - \nu^{(k)}, \quad \lambda_v^{(k)} = -2b'\lambda^{(k)} - \mu^{(k)}, \tag{7}$$

whence

$$\mu^{(k)} = -\lambda_v^{(k)} - 2b'\lambda^{(k)}, \quad \nu^{(k)} = -\lambda_u^{(k)} - 2a\lambda^{(k)}$$

Consequently we may write

$$\Delta\, d\, l^{(k)} = L^{(k)}\, du + M^{(k)}\, dv, \qquad (k = 1, 2, 3, 4), \tag{8}$$

if we put

$$\left.\begin{aligned} L^{(k)} &= d\cdot\lambda_v^{(k)} + (2\,b'\,d - A)\,\lambda^{(k)}, \\ M^{(k)} &= d'\cdot\lambda_u^{(k)} + (2\,a\,d' - B)\,\lambda^{(k)}. \end{aligned}\right\} \tag{9}$$

In order that the expressions for $d\,l^{(k)}$ may be exact differentials, we must have

$$\frac{\partial}{\partial v}\left(\frac{L^{(k)}}{\Delta}\right) = \frac{\partial}{\partial u}\left(\frac{M^{(k)}}{\Delta}\right)$$

If we make use of the equations (7) and the equations

$$\frac{\partial\Delta}{\partial u} = -4\,a\,\Delta, \quad \frac{\partial\Delta}{\partial v} = -4\,b'\,\Delta,$$

we find that these integrability conditions give rise to a system of four equations of the form

$$\begin{aligned} &(B - d'_u + 2\,a'\,d)\,\lambda_u^{(k)} - (A - d_v + 2\,b\,d')\,\lambda_v^{(k)} \\ &\quad + [-A_v + B_u + 2\,b'\,d_v - 2\,a\,d'_u - d\,(2\,a'_u + c' - 8\,a\,a') \\ &\qquad + d'\,(2\,b_v + c - 8\,b\,b') - 4\,b'A + 4\,a\,B]\,\lambda^{(k)} = 0, \\ &\qquad\qquad (k = 1, 2, 3, 4). \end{aligned} \tag{10}$$

If we interpret $y^{(1)}, \ldots, y^{(4)}$ as the homogeneous coordinates of a point P_y in space, the point P_y describes a surface as u and v pass through the values of their respective ranges. Moreover, this surface can not be a developable, for a system of equations of the form (H).† But the quantities $\lambda^{(k)}$ are the homogeneous coordinates of the plane, tangent to this surface at P_y, and therefore can not satisfy a system of equations of the form (10) with non-vanishing coefficients. For, if they did, their ratios would be functions of a single variable; *i e.*, the surface would be a developable. Consequently, the coefficients of (10) must be equal to zero. If we equate to zero the first two, we find

$$A = d_v - 2\,b\,d', \quad B = d'_u - 2\,a'\,d,$$

and these values of A and B also cause the remaining coefficient to vanish, as a consequence of the last of the integrability conditions (I).

We have obtained the following theorem:

Let $y^{(1)}, \ldots, y^{(4)}$ be four linearly independent solutions of the homogeneous system (H), and let

$$\Delta = \begin{vmatrix} y_{uv}^{(1)} & y_{uv}^{(2)} & y_{uv}^{(3)} & y_{uv}^{(4)} \\ y_u^{(1)} & y_u^{(2)} & y_u^{(3)} & y_u^{(4)} \\ y_v^{(1)} & y_v^{(2)} & y_v^{(3)} & y_v^{(4)} \\ y^{(1)} & y^{(2)} & y^{(3)} & y^{(4)} \end{vmatrix} = \sum_{k=1}^{4} y_{uv}^{(k)}\,\lambda^{(k)}, \tag{11}$$

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* *Loc. cit.*, p. 258.

† *Ibid.*, § 2.

so that $\lambda^{(k)}$ *is the co-factor of* $y_{uv}^{(k)}$ *in this determinant, which, moreover, is not identically equal to zero. If we put*

$$\left.\begin{aligned} L^{(k)} &= d\,\lambda_v^{(k)} + (2\,b\,d' + 2\,b'\,d - d_v)\,\lambda^{(k)}, \\ M^{(k)} &= d'\lambda_u^{(k)} + (2\,a\,d' + 2\,a'\,d - d'_u)\,\lambda^{(k)}, \end{aligned}\right\} \quad (k = 1, 2, 3, 4), \tag{12}$$

the quantities

$$d\,l^{(k)} = \frac{1}{\Delta}\,(L^{(k)}\,d\,u + M^{(k)}\,d\,v) \tag{13}$$

will be exact differentials, and the function

$$Y = \sum_{k=1}^{4} l^{(k)}\,y^{(k)} = \sum_{k=1}^{4} y^{(k)} \int_{(a_k,\,b_k)}^{(u,\,v)} \frac{1}{\Delta}\;(L^{(k)}\,d\,u + M^{(k)}\,d\,v), \tag{14}$$

where the lower limits of the integrals are arbitrary constants, will be a solution of the non-homogeneous system (L).

If we choose the four lower limits equal, so that

$$a_k = a, \quad b_k = b, \qquad (k = 1, 2, 3, 4),$$

we obtain that particular solution of (L) which corresponds to the initial conditions

$$Y = Y_u = Y_v = Y_{uv} = 0 \text{ for } u = a,\ v = b.$$

The quantities $\lambda^{(k)}$ satisfy a system of partial differential equations of the same type as (H). If we put

$$Y^{(k)} = \frac{\lambda^{(k)}}{\sqrt{\Delta}}, \quad (k = 1, 2, 3, 4),$$

$Y^{(1)}, \ldots., Y^{(4)}$ are solutions of the adjoined system of (H), viz.:

$$(H') \quad \left\{\begin{aligned} & Y_{uu} + 2\,a\,Y_u - 2\,b\,Y_v + (c + 2\,b_v - 4\,b\,b')\,Y = 0, \\ & Y_{vv} - 2\,a'Y_u + 2\,b'Y_v + (c' + 2\,a'_u - 4\,a\,a')\,Y = 0.^* \end{aligned}\right.$$

In terms of these quantities we find

$$\left.\begin{aligned} L^{(k)} &= \sqrt{\Delta}\,[d\,Y_v^{(k)} + (2\,bd' - d_v)\,Y^{(k)}], \\ M^{(k)} &= \sqrt{\Delta}\,[d'Y_u^{(k)} + (2\,a'd - d'_u)\,Y^{(k)}], \end{aligned}\right\} \tag{12'}$$

and these formulæ are sometimes more convenient than (12). In all applications of these formulæ it is good to remember that, since

$$\frac{\partial\,\Delta}{\partial\,u} = -4\,a\,\Delta, \quad \frac{\partial\,\Delta}{\partial\,v} = -4\,b'\,\Delta,$$

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* *Loc. cit.*, p. 259. It should be remarked that system (H') is not in general equivalent to the system obtained from (H) by taking the Riemannian adjoint of each of its equations. There are two distinct methods of generalizing the Lagrange adjoint of an ordinary linear differential equation. The above method, based on the principle of duality, may be called the geometric method, as distinguished from the analytic method of Riemann.

we shall have

$$\Delta = C e^{-4p}, \quad C = \text{const.}, \tag{15}$$

where p is determined by the conditions

$$\frac{\partial p}{\partial u} = a, \quad \frac{\partial p}{\partial v} = b',$$

which are consistent, according to the first of the integrability conditions (I). The constant C may, of course, be reduced to unity, by a proper choice of the fundamental system $y^{(1)}, \ldots, y^{(4)}$

§ 3. *Integration of System* (S) *by Series Proceeding according to Positive Integral Powers of* c_0

If the coefficients of an ordinary linear homogeneous differential equation

$$\frac{d^n y}{d x^n} + P_1 \frac{d^{n-1} y}{d x^{n-1}} + \ldots + P_n y = 0$$

are integral rational functions of a parameter μ, and if, for an initial value of the independent variable $x = x_0$ independent of μ, initial values be prescribed for $y, y', \ldots, y^{(n-1)}$ which are either independent of μ or integral (rational or transcendental) functions of μ, the corresponding solution of the differential equation will itself be an integral (rational or transcendental) function of μ. This theorem, due to Poincaré, Günther and Horn, may be applied to our system, in so far as we may by simple processes find from (S) an ordinary differential equation of the fourth order for y as a function of u, the coefficients of this equation being rational integral functions of c_0 and v, as well as of u A similar equation of course exists for y as function of v. The actual determination of the solution might be accomplished on this basis, but manifestly the lack of symmetry in the treatment accorded to the two independent variables would be a disadvantage. We therefore prefer to start anew, without making any use of the above theorem except in so far as it assures us *à priori* of the existence of solutions of the form

$$y_r = y_r^{(0)} + y_r^{(1)} c_0 + y_r^{(2)} c_0^2 + \ldots, \tag{16}$$

and even this is unnecessary, as we shall establish the existence of such solutions directly *à posteriori*.

Let us then assume for y in system (S) a development of form (16) We find at once the following conditions:

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* Poincaré, *Acta Mathematica*, Vol. IV (1884), p. 212. Günther, *Crelle's Journal*, Vol. CVII (1889) p. 312. Horn, *Mathematische Annalen*, Vol. LII (1898), p. 343.

$$\left.\begin{aligned} \frac{\partial^2 y_r^{(0)}}{\partial u^2} + 2\frac{\partial y_r^{(0)}}{\partial v} + c_1 y_r^{(0)} = 0,\\ \frac{\partial^2 y_r^{(0)}}{\partial v^2} + 2\frac{\partial y_r^{(0)}}{\partial u} + c_2 y_r^{(0)} = 0,\end{aligned}\right\} \qquad (17\text{a})$$

and

$$\left.\begin{aligned} \frac{\partial^2 y_r^{(k)}}{\partial u^2} + 2\frac{\partial y_r^{(k)}}{\partial v} + c_1 y_r^{(k)} + u\, y_r^{(k-1)} = 0,\\ \frac{\partial^2 y_r^{(k)}}{\partial v^2} + 2\frac{\partial y_r^{(k)}}{\partial u} + c_2 y_r^{(k)} + v\, y_r^{(k-1)} = 0,\end{aligned}\right\} \; (k = 1, 2, 3, \ldots .), \qquad (17\text{b})$$

which, together with the proper convergence conditions, are necessary and sufficient for the existence of a solution of (S) of the form (16).

According to (17a) we may take for $y_r^{(0)}$ the exponential $e^{\alpha_r u+\beta_r v}$, provided the constants α_r and β_r satisfy the equations

$$\left.\begin{aligned} \alpha^2 + 2\beta + c_1 = 0,\\ \beta^2 + 2\alpha + c_2 = 0.\end{aligned}\right\} \qquad (18)$$

In general, there are four linearly independent functions of this kind,

$$y_i^{(0)} = e^{\alpha_i u + \beta_i v}, \quad (i = 1, 2, 3, 4), \qquad (19)$$

and we shall, for the present, confine our discussion to this case. As we shall see at the end of this paragraph, this limitation does not essentially affect the generality of our argument and, moreover, may be easily removed afterward

The solution $y_r^{(0)} = e^{\alpha_r u+\beta_r v}$ of (17a) is characterized by the initial conditions

$$y_r^{(0)} = e^{\alpha_r a+\beta_r b}, \quad \frac{\partial y_r^{(0)}}{\partial u} = \alpha_r e^{\alpha_r a+\beta_r v}, \quad \frac{\partial y_r^{(0)}}{\partial v} = \beta_r e^{\alpha_r a+\beta_r b}, \quad \frac{\partial^2 y_r^{(0)}}{\partial u \partial v} = \alpha_r \beta_r e^{\alpha_r a+\beta_r b}$$

for $u = a$, $v = b$. We shall determine that solution of (S) which corresponds to these same initial conditions. If this solution is expressible by a series of form (16), we must satisfy the initial conditions

$$y_r^{(k)} = \frac{\partial y_r^{(k)}}{\partial u} = \frac{\partial y_r^{(k)}}{d v} = \frac{\partial^2 y_r^{(k)}}{\partial u \partial v} = 0, \qquad k \geqq 1, \qquad (20)$$

for $u = a$, $v = b$, while

$$y_r^{(0)} = e^{\alpha_r u + \beta_r v}$$

We proceed to calculate $y_r^{(k)}$ by applying the method of § 2 to (17b) In the notation of § 2 we have

$$\begin{aligned} a = 0, \quad b = 1, \quad c = c_1, \quad d = u\, y_r^{(k-1)},\\ a' = 1, \quad b' = 0, \quad c' = c_2, \quad d' = v\, y_r^{(k-1)}\end{aligned}$$

Of the five integrability conditions (I), the first four are obviously satisfied. The left member of the last reduces to

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* A complete discussion of the various cases is to be found in my paper "On a Certain Class of Self-Projective Surfaces," to which I have already referred.

$$v\left(\frac{\partial^2 y_r^{(k-1)}}{\partial u^2} + 2\frac{\partial y_r^{(k-1)}}{\partial v} + c_1 y_r^{(k-1)}\right) - u\left(\frac{\partial^2 y_r^{(k-1)}}{\partial v^2} + 2\frac{\partial y_r^{(k-1)}}{d u} + c_2 y_r^{(k-1)}\right),$$

and this is equal to zero on account of (17b), if $k \geqq 2$, and on account of (17a), if $k = 1$.

The homogeneous system, which corresponds to (17b), is (17a), and its solutions are determined by (18) and (19) Therefore, $\alpha_1, \ldots, \alpha_4$ are the four roots of the equation

$$\alpha^4 + 2c_1\alpha^2 + 8\alpha + c_1^2 + 4c_2 = 0, \tag{21}$$

so that

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = \beta_1 + \beta_2 + \beta_3 + \beta_4 = 0$$

Consequently we find

$$y_1^{(0)}\, y_2^{(0)}\, y_3^{(0)}\, y_4^{(0)} = 1$$

and

$$\Delta = \begin{vmatrix} \alpha_1\beta_1 & \alpha_2\beta_2 & \alpha_3\beta_3 & \alpha_4\beta_4 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ 1 & 1 & 1 & 1 \end{vmatrix} = \frac{1}{4}\begin{vmatrix} \alpha_1^3 & \alpha_2^3 & \alpha_3^3 & \alpha_4^3 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \alpha_1^2 & \alpha_2^2 & \alpha_3^2 & \alpha_4^2 \\ 1 & 1 & 1 & 1 \end{vmatrix}, \tag{22}$$

or

$$4\Delta = -(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)(\alpha_1-\alpha_4)(\alpha_2-\alpha_3)(\alpha_2-\alpha_4)(\alpha_3-\alpha_4)$$

We then find, for the co-factor of $\dfrac{\partial^2 y_i^{(0)}}{\partial u \partial v}$ in Δ, the value

$$\lambda^{(i)} = \frac{k_i}{2 y_i^{(0)}}, \qquad (i = 1, 2, 3, 4),$$

where

$$\left.\begin{aligned} k_1 &= (\alpha_2-\alpha_3)(\alpha_2-\alpha_4)(\alpha_3-\alpha_4), & k_3 &= (\alpha_4-\alpha_1)(\alpha_4-\alpha_2)(\alpha_1-\alpha_2), \\ k_2 &= -(\alpha_3-\alpha_4)(\alpha_3-\alpha_1)(\alpha_4-\alpha_1), & k_4 &= -(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)(\alpha_2-\alpha_3) \end{aligned}\right\} \tag{23}$$

If we substitute these values in (12), we find

$$L_{r,i}^{(k)} = \frac{k_i}{2} e^{-\alpha_i u - \beta_i v}\left[(-\beta_i u + 2v)\, y_r^{(k-1)} - u\frac{\partial y_r^{(k-1)}}{\partial v}\right],$$

$$M_{r,i}^{(k)} = \frac{k_i}{2} e^{-\alpha_i u - \beta_i v}\left[(2u - \alpha_i v)\, y_r^{(k-1)} - v\frac{\partial y_r^{(k-1)}}{\partial u}\right],$$

where we have introduced a slight change of notation as compared with (12), the upper index k of (12) having been replaced by the lower index i, while the significance of the two new indices r and k, which appear in the last two equations, is obvious.

From the above values of Δ and k_1 we obtain the value of their quotient,

$$\frac{4\Delta}{k_1} = -(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)(\alpha_1-\alpha_4)$$

But, on account of (21), we have the relations

$$\Sigma \alpha_i = 0, \quad \Sigma \alpha_i \alpha_j = 2 c_1, \quad \Sigma \alpha_i \alpha_j \alpha_k = -8, \quad \alpha_1 \alpha_2 \alpha_3 \alpha_4 = c_1^2 + 4 c_2,$$

so that

$$\frac{4 \Delta}{k_1} = -4 (\alpha_1^3 + c_1 \alpha_1 + 2) = 8 (\alpha_1 \beta_1 - 1),$$

proving the first of the four equations

$$\frac{\Delta}{k_i} = 2 (\alpha_i \beta_i - 1), \qquad (i = 1, 2, 3, 4) \tag{24}$$

Clearly, $\alpha_i \beta_i - 1$ will be different from zero, since α_i is a simple root of (21)

From (14) we find, finally,

$$y_r^{(k)} = \sum_{i=1}^{4} e^{\alpha_i u + \beta_i v} \int_{a,\, b}^{u,\, v} \frac{e^{-\alpha_i u - \beta_i v}}{4 (\alpha_i \beta_i - 1)} \left[\left\{ (-\beta_i u + 2 v)\, y_r^{(k-1)} - u \frac{\partial y_r^{(k-1)}}{\partial v} \right\} d u \right.$$
$$\left. + \left\{ (2 u - \alpha_i v)\, y_r^{(k-1)} - v \frac{\partial y_r^{(k-1)}}{\partial u} \right\} d v \right] \tag{25}$$

The lower limits of the four integrals which appear in this formula have been chosen equal to (a, b) so as to satisfy the conditions (20) That the expression (25) for $y_r^{(k)}$ actually satisfies these conditions may be seen as follows. We may write (25) in the form

$$y_r^{(k)} = \sum_{i=1}^{4} l_{r,i}^{(k)} y_i^{(0)}, \tag{25a}$$

where the quantities $l_{r,i}^{(k)}$ have been determined, in accordance with the method of § 2, in such a way that

$$\frac{\partial y_r^{(k)}}{\partial u} = \sum_{i=1}^{4} l_{r,i}^{(k)} \frac{\partial y_i^{(0)}}{\partial u}, \quad \frac{\partial y_r^{(k)}}{\partial v} = \sum_{i=1}^{4} l_{r,i}^{(k)} \frac{\partial y_i^{(0)}}{\partial v}, \quad \frac{\partial^2 y_r^{(k)}}{\partial u \partial v} = \sum_{i=1}^{4} l_{r,i}^{(k)} \frac{\partial^2 y_i^{(0)}}{\partial u \partial v}$$

Consequently, all of these expressions will actually reduce to zero for $u = a$, $v = b$

In order to establish the convergence of the expansion (16) for y_r, it will be convenient to write

$$\left. \begin{aligned} K_i (u, v;\, p, q) &= \frac{e^{\alpha_i (u-p) + \beta_i (v-q)}}{8 (\alpha_i \beta_i - 1)}, \\ S_i &= \int_{a,\, b}^{p,\, q} d\, (-\beta_i p^2 + 4 p q - \alpha_i q^2) \end{aligned} \right\} \quad (i = 1, 2, 3, 4), \tag{26}$$

Then we shall have

$$\left.\begin{aligned}
y_r^{(k)} &= \int_{a,\,b}^{u,\,v} \sum_{i=1}^{4} K_i \left\{ y_r^{(k-1)}\, dS_i - \frac{\partial y_r^{(k-1)}}{\partial q}\, d\,(p^2) - \frac{\partial y_r^{(k-1)}}{\partial p}\, d\,(q^2) \right\}; \\
\frac{\partial y_r^{(k)}}{\partial u} &= \int_{a,\,b}^{u,\,v} \sum_{i=1}^{4} \alpha_i K_i \left\{ y_r^{(k-1)}\, dS_i - \frac{\partial y_r^{(k-1)}}{\partial q}\, d\,(p^2) - \frac{\partial y_r^{(k-1)}}{\partial p}\, d\,(q^2) \right\}, \\
\frac{\partial y_r^{(k)}}{\partial v} &= \int_{a,\,b}^{u,\,v} \sum_{i=1}^{4} \beta_i K_i \left\{ y_r^{(k-1)}\, dS_i - \frac{\partial y_r^{(k-1)}}{\partial q}\, d\,(p^2) - \frac{\partial y_r^{(k-1)}}{\partial p}\, d\,(q^2) \right\}, \\
\frac{\partial^2 y_r^{(k)}}{\partial u\, \partial v} &= \int_{a,\,b}^{u,\,v} \sum_{i=1}^{4} \alpha_i \beta_i K_i \left\{ y_r^{(k-1)}\, dS_i - \frac{\partial y_r^{(k-1)}}{d q}\, d\,(p^2) - \frac{\partial y_r^{(k-1)}}{\partial p}\, d\,(q^2) \right\}.
\end{aligned}\right\} \qquad (27)$$

A simple induction proof, involving the use of equation (25), shows that, for every value of k, $y_r^{(k)}$ and all of its partial derivatives are continuous (and even analytic) functions of u and v, for all finite values of these variables. The path of integration from (a, b) to (u, v) may be chosen arbitrarily in the finite portion of the plane, since all of the differentials involved in (25) are exact. We may, moreover, without any essential restriction of generality, assume $a = b = 0$.

Let N be a positive number not less than the largest value assumed by $|e^{\alpha_r p + \beta_r q}|$ on the path of integration. We shall then have

$$|y_r^0| \leqq N, \quad \left| \frac{\partial y_r^0}{\partial p} \right| \leqq |\alpha_r|\, N, \quad \left| \frac{\partial y_r^{(0)}}{\partial q} \right| \leqq |\beta_r|\, N.$$

For any assigned finite values of u and v, there will exist a positive constant M_i, such that, for all values of p and q along the path of integration,

$$|K_i\,(u, v;\, p, q)| \leqq M_i.$$

Let

$$M = \sum_{i=1}^{4} M_i,$$

and denote by α and β two positive numbers, such that

$$|\alpha_i| \leqq \alpha, \quad |\beta_i| \leqq \beta, \qquad (i = 1, 2, 3, 4),$$

and consequently,

$$\left| \sum_{i=1}^{4} K_i\,(u, v;\, p, q) \right| \leqq M, \quad \left| \sum_{i=1}^{4} \alpha_i K_i \right| \leqq \alpha M, \quad \left| \sum_{i=1}^{4} \beta_i K_i \right| \leqq \beta M.$$

Introduce three positive functions by putting

$$P = \int_{0,\,0}^{p,\,q} |d\,(p^2)|, \quad Q = \int_{0,\,0}^{p,\,q} |d(q^2)|, \quad S = \int_{0,\,0}^{p,\,q} d\,(\beta\,|p|^2 + 4\,|p|\,.\,|q| + \alpha\,|q|^2),$$

and we shall have

$$|dS_i| \leqq dS, \qquad (i = 1, 2, 3, 4)$$

If now we put $k = 1$ in (27), we find

$$|y_r^{(1)}| \leqq \int_{0,0}^{u,v} \sum_{i=1}^{4} M_i \{N\, dS + \beta N\, dP + \alpha N\, dQ\} \leqq NMT,$$

where

$$T = S + \beta P + \alpha Q,$$

and similarly,

$$\left|\frac{\partial y_r^{(1)}}{\partial u}\right| \leqq \alpha NMT, \quad \left|\frac{\partial y_r^{(1)}}{\partial v}\right| \leqq \beta NMT, \quad \left|\frac{\partial^2 y_r^{(1)}}{\partial u\, \partial v}\right| \leqq \alpha\beta NMT.$$

If we make use of these inequalities after putting $k = 2$ in (27), we find

$$|y_r^{(2)}| \leqq N\frac{M^2 T^2}{2}, \text{ etc.},$$

and, by induction, in general,

$$|y_r^{(k)}| \leqq N\frac{M^k T^k}{k!}, \quad \left|\frac{\partial y_r^{(k)}}{\partial u}\right| \leqq \alpha N\frac{M^k T^k}{k!}, \quad \left|\frac{\partial y_r^{(k)}}{\partial v}\right| \leqq \beta N\frac{M^k T^k}{k!}, \text{ etc.}$$

These inequalities assure us of the absolute and uniform convergence of our series for y_r, and of its term-by-term differentiability with respect to u and v taken either separately or together. It is easy to show, further, that $\frac{\partial^2 y_r}{\partial u^2}$ and $\frac{\partial^2 y_r}{\partial v^2}$ may also be obtained by repeated term-by-term differentiations from y_r as absolutely and uniformly convergent series. Consequently, we have obtained the following theorem:

The series

$$y_r = \sum_{k=0}^{\infty} y_r^{(k)} c_0^k,$$

for which $y_r^{(0)} = e^{\alpha_r u + \beta_r v}$, *and whose general term is determined by the recursion formula* (25), *is absolutely and uniformly convergent for all finite values of* u, v *and* c_0, *and represents that solution of the completely integrable system* (S), *whose initial values for* $u = a$, $v = b$ *coincide with those of the exponential function* $e^{\alpha_r u + \beta_r v}$

We have computed the solutions y_r under the assumption that the four roots $\alpha_1, \ldots, \alpha_4$ of (21) are distinct. But this assumption, as we have already stated, does not really restrict the generality of our method. For, by means of a linear transformation of the independent variables

$$u = \bar{u} + h, \quad v = \bar{v} + k, \tag{28a}$$

system (S) may be transformed into a new system of the same kind, whose constants have the values

$$\bar{c}_0 = c_0, \quad \bar{c}_1 = c_1 + h c_0, \quad \bar{c}_2 = c_2 + k c_0. \qquad (28\text{b})$$

If $c_0 \neq 0$, we may therefore transform (S) into another system of the same form whose constants $\bar{c}_1$ and $\bar{c}_2$ have any desired values, for instance $\bar{c}_1 = \bar{c}_2 = 0$, in which case the four roots of (21) will actually be distinct. On the other hand, it would not be difficult to develop formulæ to take the place of (25) in the case of two or more equal roots.

The linear transformation (28a) enables us to materially simplify the appearance of the differential equations (S), but we have made no use of this transformation in developing our theory up to this point, for two reasons. In the first place, there would have resulted no essential simplification in any of our formulæ sufficient to make up for the corresponding loss of symmetry. But, in the second place, the general formulæ have a real significance which might otherwise have escaped our notice, and which gives us a real insight into the nature of the integral surface.

In fact, the linear transformation (28a) may be interpreted as a change of position on the integral surface, instead of as a change of the parameters of reference. Now, if c_0 is small, our formulæ show that the surface whose parametric equations are

$$y_r = e^{\alpha_r u + \beta_r v}, \qquad (r = 1, 2, 3, 4), \qquad (29)$$

may be regarded as projectively equivalent to a first approximation of our integral surface in the neighborhood of the point for which $u = a$, $v = b$. If $\bar{\alpha}_r$ and $\bar{\beta}_r$ are the values of α_r and β_r which correspond to the constants $\bar{c}_1$ and $\bar{c}_2$, the surface

$$y_r = e^{\bar{\alpha}_r \bar{u} + \bar{\beta}_r \bar{v}}, \qquad (r = 1, 2, 3, 4),$$

will, in the same way, be projectively equivalent to a first approximation of the integral surface in the vicinity of the point $u = a + h$, $v = b + k$. In other words, for small values of c_0, an integral surface Σ of (S) has associated with each of its points P, a self-projective surface of the form (29), whose exponents α_r, β_r change with P, and which may be regarded as a first approximation of Σ in the vicinity of P.

One further remark is of importance for our later developments. From the nature of our convergence proof by dominant functions, it is clear that the series

$$y_r^{(0)} + y_r^{(1)} c_0 + y_r^{(2)} c_0^2 + \ldots.$$

will still be a uniformly and absolutely convergent series for all finite values of u, v and c_0, if the recursion formula (25) be altered by writing in place of

---

* Cf. foot-note on page 239.

(a, b), the common lower limit of all of the integrals, a different lower limit $(a_i^{(k)}, b_i^{(k)})$ for each of the integrals, provided that all of these limits are finite. Moreover, the function of u and v thus defined will still be a solution of (S), corresponding, however, to altered initial conditions.

§ 4. *Properties of the Coefficients of the Series Obtained in the Preceding Section.*

Let us make the generalization referred to at the end of the preceding section, and on that assumption calculate $y_r^{(1)}$. If we make use of the abbreviation involved in writing (25) in the form (25a), we find

$$l_{r,r}^{(1)} = \frac{1}{4(\alpha_r\beta_r-1)}\Big[-\beta_r u^2 + 2uv - \alpha_r v^2\Big]_{(a_r^{(1)},\, b_r^{(1)})}^{(u,\,v)},$$

$$l_{r,i}^{(1)} = \left[\frac{e^{(\alpha_r-\alpha_i)u+(\beta_r-\beta_i)v}}{2(\alpha_i\beta_i-1)(\alpha_r-\alpha_i)(\beta_r-\beta_i)}\{(\alpha_r-\alpha_i)u+(\beta_r-\beta_i)v-1\}\right]_{(a_i^{(1)},\, b_i^{(1)})}^{(u,\,v)}, \quad i \neq r.$$

Then

$$\begin{aligned} y_r^{(1)} = {} & e^{\alpha_r u+\beta_r v}\left[\frac{-\beta_r u^2+2uv-\alpha_r v^2}{4(\alpha_r\beta_r-1)} + \sum_{i=1}^{4}\frac{(\alpha_r-\alpha_i)u+(\beta_r-\beta_i)v-1}{2(\alpha_i\beta_i-1)(\alpha_r-\alpha_i)(\beta_r-\beta_i)}\right] \\ & -\sum_{i=1}^{4}{}' e^{\alpha_i u+\beta_i v}\frac{e^{(\alpha_r-\alpha_i)a_i^{(1)}(\beta_r-\beta_i)b_i^{(1)}}}{2(\alpha_i\beta_i-1)(\alpha_r-\alpha_i)(\beta_r-\beta_i)}\{(\alpha_r-\alpha_i)a_i^{(1)}+(\beta_r-\beta_i)b_i^{(1)}-1\} \\ & - e^{\alpha_r u+\beta_r v}\frac{-\beta_r a_r^{(1)2}+2a_r^{(1)}b_r^{(1)}-\alpha_r b_r^{(1)2}}{4(\alpha_r\beta_r-1)}, \end{aligned} \tag{30}$$

where the stroke on the summation symbol indicates that the value $i = r$ is to be omitted. It will obviously be possible to find finite numbers $a_i^{(1)}$, $b_i^{(1)}$ and $a_r^{(1)}$, $b_r^{(1)}$ so as to cause the terms $e^{\alpha_i u+\beta_i v}$ $(i \neq r)$ to disappear, and to reduce to zero the constant term of the expression which is multiplied by $e^{\alpha_r u+\beta_r v}$ If this be done, we have

$$\begin{aligned} y_r^{(1)} &= e^{\alpha_r u+\beta_r v}\left[\frac{-\beta_r u^2+2uv-\alpha_r v^2}{4(\alpha_r\beta_r-1)} + \sum_{i=1}^{4}{}'\frac{(\alpha_r-\alpha_i)u+(\beta_r-\beta_i)v}{2(\alpha_i\beta_i-1)(\alpha_r-\alpha_i)(\beta_r-\beta_i)}\right] \\ &= e^{\alpha_r u+\beta_r v}P_r^{(1)}(u, v), \end{aligned} \tag{30a}$$

so that $y_r^{(1)}$ reduces to an exponential factor multiplied by a polynomial $P_r^{(1)}(u, v)$ of the second degree in u and v, whose constant term is equal to zero.

We may generalize this result. For every value of k there exists a solution $y_r^{(k)}$ of (17b), expressible in the form

$$y_r^{(k)} = e^{\alpha_r u+\beta_r v}\sum_{j=1}^{2k} A_{r,j}^{(k)} = e^{\alpha_r u+\beta_r v}P_r^{(k)}(u, v),$$

where $A_{r,j}^{(k)}$ is a homogeneous polynomial of degree j in u and v. In other words, $y_r^{(k)}$ is equal to $e^{\alpha_r u+\beta_r v}$ multiplied by a polynomial of degree $2k$ in u and v, without a constant term.

The theorem is true for $k=1$. We proceed to prove the general theorem by induction. Let us suppose, therefore, that

$$y_r^{(k-1)} = e^{\alpha_r u+\beta_r v} \sum_{j=1}^{2k-2} A_{r,j}^{(k-1)},$$

and consequently

$$\frac{\partial y_r^{(k-1)}}{\partial u} = e^{\alpha_r u+\beta_r v} \sum_{j=1}^{2k-2} \left(\frac{\partial A_{r,j}^{(k-1)}}{\partial u} + \alpha_r A_{r,j}^{(k-1)}\right),$$

$$\frac{\partial y_r^{(k-1)}}{\partial v} = e^{\alpha_r u+\beta_r v} \sum_{j=1}^{2k-2} \left(\frac{\partial A_{r,j}^{(k-1)}}{\partial v} + \beta_r A_{r,j}^{(k-1)}\right).$$

We shall find

$$l_{r,i}^{(k)} = \int_{a_i^{(k)},\, b_i^{(k)}}^{(u,v)} e^{(\alpha_r-\alpha_i)u+(\beta_r-\beta_i)v} \left[\{P_{r,i,1}^{(k-1)} + P_{r,i,2}^{(k-1)} + \dots + P_{r,i,2k-1}^{(k-1)}\}\, du + \{Q_{r,i,1}^{(k-1)} + \dots + Q_{r,i,2k-1}^{(k-1)}\}\, dv\right], \qquad (31)$$

where

$$\left.\begin{aligned} P_{r,i,j}^{(k-1)} &= \frac{1}{4(\alpha_i\beta_i-1)}\left[\{-(\beta_r+\beta_i)u+2v\}\, A_{r,j-1}^{(k-1)} - u\frac{\partial A_{r,j}^{(k-1)}}{\partial v}\right], \\ Q_{r,i,j}^{(k-1)} &= \frac{1}{4(\alpha_i\beta_i-1)}\left[\{2u-(\alpha_r+\alpha_i)v\}\, A_{r,j-1}^{(k-1)} - v\frac{\partial A_{r,j}^{(k-1)}}{\partial u}\right], \\ &(j=1,2,3,\dots,2k-1), \end{aligned}\right\} \qquad (32)$$

in the application of which formulæ we must remember to put

$$A_{r\,0}^{(k-1)} = A_{2k-1}^{(k-1)} = 0.$$

Clearly, $P_{r,i,j}^{(k-1)}$ and $Q_{r,i,j}^{(k-1)}$ are homogeneous polynomials in u and v, of degree j. From our general theory we know that the expression under the integral sign in (31) is an exact differential. We must, therefore, have

$$\left.\begin{aligned} (\beta_r-\beta_i)\, P_{r,i,2k-1}^{(k-1)} &= (\alpha_r-\alpha_i)\, Q_{r,i,2k-1}^{(k-1)}, \\ (\beta_r-\beta_i)\, P_{r,i,2k-2}^{(k-1)} + \frac{\partial P_{r,i,2k-1}^{(k-1)}}{\partial v} &= (\alpha_r-\alpha_i)\, Q_{r,i,2k-2}^{(k-1)} + \frac{\partial Q_{r,i,2k-1}^{(k-1)}}{\partial u}, \\ \cdots\cdots\cdots&\cdots\cdots\cdots, \\ (\beta_r-\beta_i)\, P_{r,i,1}^{(k-1)} + \frac{\partial P_{r,i,2}^{(k-1)}}{\partial v} &= (\alpha_r-\alpha_i)\, Q_{r,i,1}^{(k-1)} + \frac{\partial Q_{r,i,2}^{(k-1)}}{\partial u}, \\ \frac{\partial P_{r,i,1}^{(k-1)}}{\partial v} &= \frac{\partial Q_{r,i,1}^{(k-1)}}{\partial u}. \end{aligned}\right\} \qquad (33)$$

Consider first the case $i \neq r$. We may write

$$l_{r,i}^{(k)} = \left[e^{(\alpha_r-\alpha_i)u+(\beta_r-\beta_i)v}\{R_{r,i,0}^{(k)} + R_{r,i,1}^{(k)} + \dots + R_{r,i,2k-1}^{(k)}\}\right]_{a_i^{(k)},\, b_i^{(k)}}^{u,\,v},$$

where $R_{r,i,j}^{(k)}$ is a homogeneous polynomial of degree j in u and v. In fact, in order that this may be so, it is necessary and sufficient to satisfy the conditions

$$
\begin{aligned}
&\frac{\partial R^{(k)}_{r,i,1}}{\partial u}+(\alpha_r-\alpha_i)\,R^{(k)}_{r,i,0}=0, &&\frac{\partial R^{(k)}_{r,i,1}}{\partial v}+(\beta_r-\beta_i)\,R^{(k)}_{r,i,0}=0,\\
&\frac{\partial R^{(k)}_{r,i,2}}{\partial u}+(\alpha_r-\alpha_i)\,R^{(k)}_{r,i,1}=P^{(k-1)}_{r,i,1}, &&\frac{\partial R^{(k)}_{r,i,2}}{\partial v}+(\beta_r-\beta_i)\,R^{(k)}_{r,i,1}=Q^{(k-1)}_{r,i,1},\\
&\qquad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, &&\qquad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
&\frac{\partial R^{(k)}_{r,i,2k-1}}{\partial u}+(\alpha_r-\alpha_i)\,R^{(k)}_{r,i,2k-2}=P^{(k-1)}_{r,i,2k-2}, &&\frac{\partial R^{(k)}_{r,i,2k-1}}{\partial v}+(\beta_r-\beta_i)\,R^{(k)}_{r,i,2k-2}=Q^{(k-1)}_{r,i,2k-2},\\
&\qquad(\alpha_r-\alpha_i)\,R^{(k)}_{r,i,2k-1}=P^{(k-1)}_{r,i,2k-1}, &&\qquad(\beta_r-\beta_i)\,R^{(k)}_{r,i,2k-1}=Q^{(k-1)}_{r,i,2k-1},
\end{aligned}
$$

which are consistent on account of (33) and give

$$
\begin{aligned}
R^{(k)}_{r,i,2k-1}&=\frac{P^{(k-1)}_{r,i,2k-1}}{\alpha_r-\alpha_i}=\frac{Q^{(k-1)}_{r,i,2k-1}}{\beta_r-\beta_i},\\
R^{(k)}_{r,i,2k-2}&=\frac{P^{(k-1)}_{r,i,2k-2}}{\alpha_r-\alpha_i}-\frac{\dfrac{\partial P^{(k-1)}_{r,i,2k-1}}{\partial u}}{(\alpha_r-\alpha_i)^2}=\frac{Q^{(k-1)}_{r,i,2k-2}}{\beta_r-\beta_i}-\frac{\dfrac{\partial Q^{(k-1)}_{r,i,2k-1}}{\partial v}}{(\beta_r-\beta_i)^2}\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
R^{(k)}_{r,i,2k-\lambda}&=\frac{P^{(k-1)}_{r,i,2k-\lambda}}{\alpha_r-\alpha_i}-\frac{\dfrac{\partial P^{(k-1)}_{r,i,2k-\lambda+1}}{\partial u}}{(\alpha_r-\alpha_i)^2}+\ldots\ldots+(-1)^{\lambda-1}\frac{\dfrac{\partial^{\lambda-1}P^{(k-1)}_{r,i,2k-1}}{\partial u^{\lambda-1}}}{(\alpha_r-\alpha_i)^{\lambda}}\\
&=\frac{Q^{(k-1)}_{r,i,2k-\lambda}}{\beta_r-\beta_i}-\frac{\dfrac{\partial Q^{(k-1)}_{r,i,2k-\lambda+1}}{\partial v}}{(\beta_r-\beta_i)^2}+\ldots\ldots+(-1)^{\lambda-1}\frac{\dfrac{\partial^{\lambda-1}Q^{(k-1)}_{r,i,2k-1}}{\partial v^{\lambda-1}}}{(\beta_r-\beta_i)^{\lambda}},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}
$$

If we write $2k-\lambda=j$, we obtain the general formula

$$
R^{(k)}_{r,i,j}=\sum_{\lambda=0}^{2k-j-1}\frac{(-1)^{\lambda}}{(\alpha_r-\alpha_i)^{\lambda+1}}\frac{\partial^{\lambda}P^{(k-1)}_{r,i,j+\lambda}}{\partial u^{\lambda}}=\sum_{\lambda=0}^{2k-j-1}\frac{(-1)^{\lambda}}{(\beta_r-\beta_i)^{\lambda+1}}\frac{\partial^{\lambda}Q^{(k-1)}_{r,i,j+\lambda}}{\partial v^{\lambda}}, \tag{34}
$$

$$
(j=0,1,2,\ldots.,2k-1),\quad (i=1,2,3,4;\ i\neq r),
$$

which gives, upon substituting the values (32), the following two equivalent expressions:

$$
\left.
\begin{aligned}
R^{(k)}_{r,i,j}&=\sum_{\lambda=0}^{2k-j-1}\frac{(-1)^{\lambda}}{4(\alpha_r-\alpha_i)^{\lambda+1}(\alpha_i\beta_i-1)}\Bigg[\}-\beta_r+\beta_i)u+2v\}\,\frac{\partial^{\lambda}A^{(k-1)}_{r,j+\lambda-1}}{\partial u^{\lambda}}\\
&\qquad-\lambda(\beta_r+\beta_i)\,\frac{\partial^{\lambda-1}A^{(k-1)}_{r,j+\lambda-1}}{\partial u^{\lambda-1}}-u\,\frac{\partial^{\lambda+1}A^{(k-1)}_{r,j+\lambda}}{\partial u^{\lambda}\,\partial v}-\lambda\,\frac{\partial^{\lambda}A^{(k-1)}_{r,j+\lambda}}{\partial u^{\lambda-1}\,\partial v}\Bigg],\\
R^{(k)}_{r,i,j}&=\sum_{\lambda=0}^{2k-j-1}\frac{(-1)^{\lambda}}{4(\beta_r-\beta_i)^{\lambda+1}(\alpha_i\beta_i-1)}\Bigg[\{2u-(\alpha_r+\alpha_i)\,v\}\,\frac{\partial^{\lambda}A^{(k-1)}_{r,j+\lambda-1}}{\partial v^{\lambda}}\\
&\qquad-\lambda(\alpha_r+\alpha_i)\,\frac{\partial^{\lambda-1}A^{(k-1)}_{r,j+\lambda-1}}{\partial v^{\lambda-1}}-v\,\frac{\partial^{\lambda+1}A^{(k-1)}_{r,j+\lambda}}{\partial u\,\partial v^{\lambda}}-\lambda\,\frac{\partial^{\lambda}A^{(k-1)}_{r,j+\lambda}}{\partial u\,\partial v^{\lambda-1}}\Bigg],\\
&\qquad(j=0,1,2,\ldots.,2k-1),\quad (i=1,2,3,4;\ i\neq r).
\end{aligned}
\right\} \tag{35}
$$

For $i = r$, we find

$$l_{r,r}^{(k)} = \int_{(a_r^{(k)},\, b_r^{(k)})}^{(u,\, v)} [\{P_{r,r,1}^{(k-1)} + \dots + P_{r,r,2k-1}^{(k-1)}\}\, du + \{Q_{r,r,1}^{(k-1)} + \dots + Q_{r,r,2k-1}^{(k-1)}\}\, dv],$$

and the conditions (33) reduce to

$$\frac{\partial P_{r,r,j}^{(k-1)}}{\partial v} = \frac{\partial Q_{r,r,j}^{(k-1)}}{\partial u}, \qquad (j = 1, 2, \dots, 2k-1).$$

We may write

$$l_{r,r}^{(k)} = [R_{r,r,0}^{(k)} + R_{r,r,1}^{(k)} + \dots + R_{r,r,2k}^{(k)}]_{(a_r^{(k)},\, b_r^{(k)})}^{(u,\, v)},$$

where $R_{r,r,j+1}^{(k)}$ is a homogeneous polynomial of order $j+1$, subject to the compatible conditions

$$\frac{\partial R_{r,r,j+1}^{(k)}}{\partial u} = P_{r,r,j}^{(k-1)} = \frac{1}{4(\alpha_r \beta_r - 1)} \left[\frac{\partial q_r}{\partial u} A_{r,j-1}^{(k-1)} - u \frac{\partial A_{r,j}^{(k-1)}}{\partial v}\right],$$

$$\frac{\partial R_{r,r,j+1}^{(k)}}{\partial v} = Q_{r,r,j}^{(k-1)} = \frac{1}{4(\alpha_r \beta_r - 1)} \left[\frac{\partial q_r}{\partial v} A_{r,j-1}^{(k-1)} - v \frac{\partial A_{r,j}^{(k-1)}}{\partial u}\right],$$

$$(j = 1, 2, 3, \dots, 2k-1),$$

where

$$q_r = -\beta_r u^2 + 2uv - \alpha_r v^2. \tag{36}$$

If we multiply both members of the first equation by u, and those of the second by v, and add, making use of Euler's theorem on homogeneous functions, we find

$$(j+1) R_{r,r,j+1}^{(k)} = \frac{1}{4(\alpha_r \beta_r - 1)} \left[2 q_r A_{r,j-1}^{(k-1)} - \left(v^2 \frac{\partial A_{r,j}^{(k-1)}}{\partial u} + u^2 \frac{\partial A_{r,j}^{(k-1)}}{\partial v}\right)\right], \tag{37}$$

$$(j = 1, 2, \dots, 2k-1),$$

to which we may add

$$R_{r,r,1}^{(k)} = 0. \tag{37a}$$

The value of the arbitrary constant $R_{r,r,0}^{(k)}$ is immaterial. The resulting expression for $y_r^{(k)}$ will be

$$\begin{aligned} y_r^{(k)} = {} & e^{\alpha_r u + \beta_r v} [R_{r,r,2}^{(k)} + \dots + R_{r,r,2k}^{(k)}]_{a_r^{(k)},\, b_r^{(k)}}^{u,\, v} \\ & + e^{\alpha_r u + \beta_r v} \sum_{i=1}^{4}{}' \{R_{r,i,0}^{(k)} + \dots + R_{r,i,2k-1}^{(k)}\} \\ & - \sum_{i=1}^{4}{}' e^{\alpha_i u + \beta_i v + (\alpha_r - \alpha_i) a_i^{(k)} + (\beta_r - \beta_i) b_i^{(k)}} [R_{r,i,0}^{(k)} + \dots + R_{r,i,2k-1}^{(k)}]_{a_i^{(k)},\, b_i^{(k)}}, \end{aligned} \tag{38}$$

and this will have the desired form, if and only if the limits are chosen in such a way that

$$\left.\begin{aligned} & [R_{r,i,0}^{(k)} + \dots + R_{r,i,2k-1}^{(k)}]_{a_i^{(k)},\, b_i^{(k)}} = 0, \\ & [R_{r,r,2}^{(k)} + \dots + R_{r,r,2k}^{(k)}\ ]_{a_r^{(k)},\, b_r^{(k)}} = \sum_{i=1}^{4}{}' R_{r,i,0}^{(k)}. \end{aligned}\right\} (i = 1, 2, 3, 4;\ i \neq r), \tag{39}$$

For every finite value of k this can certainly be done, and we shall then have the following formula for $y_r^{(k)}$:

$$y_r^{(k)} = e^{\alpha_r u + \beta_r v} [R^{(k)}_{r, r, 2k} + \sum_{i=1}^{4} (R^{(k)}_{r, i, 1} + \dots + R^{(k)}_{r, i, 2k-1})],$$

if we remember that, according to (37a),

$$R^{(k)}_{r, r, 1} = 0.$$

Therefore,

$$A^{(k)}_{r, j} = \sum_{i=1}^{4} R^{(k)}_{r, i, j}, \quad (j = 1, 2, \dots, 2k-1), \quad A^{(k)}_{r, 2k} = R^{(k)}_{r, r, 2k},$$

whence, making use of (35) and (37),

$$A^{(k)}_{r, j} = \frac{1}{4j(\alpha_r \beta_r - 1}\left[2 q_r A^{(k-1)}_{r, j-2} - \left(v^2 \frac{\partial A^{(k-1)}_{r, j-1}}{\partial u} + u^2 \frac{\partial A^{(k-1)}_{r, j-1}}{\partial v}\right)\right]$$
$$+ \frac{1}{4} \sum_{i=1}^{4}{}' \sum_{\lambda=0}^{2k-j-1} \frac{(-1)^\lambda}{(\alpha_r - \alpha_i)^{\lambda+1}(\alpha_i \beta_i - 1)}\left[\{-(\beta_r + \beta_i)u + 2v\} \frac{\partial^\lambda A^{(k-1)}_{r, j+\lambda-1}}{\partial u^\lambda}\right.$$
$$\left. - \lambda(\beta_r + \beta_i) \frac{\partial^{\lambda-1} A^{(k-1)}_{r, j+\lambda-1}}{\partial u^{\lambda-1}} - u \frac{\partial^{\lambda+1} A^{(k-1)}_{r, j+\lambda}}{\partial u^\lambda \partial v} - \lambda \frac{\partial^\lambda A^{(k-1)}_{r, j+\lambda}}{\partial u^{\lambda-1} \partial v}\right],$$
$$(j = 1, 2, \dots, 2k-1).$$

$$A^{(k)}_{r, 2k} = \frac{q_r}{4k(\alpha_r \beta_r - 1)} A^{(k-1)}_{r, 2k-2}.$$

A second equivalent form for $A^{(k)}_{r, j}$ may be found by using the second of the two equations (35).

From the last formula we find at once

$$A^{(k)}_{2k} = \frac{q_r^k}{4^k k! (\alpha_r \beta_r - 1)^k}, \qquad (\lambda = 1, 2, 3, \dots). \tag{40}$$

Let us put

$$s_{r, \lambda} = \sum_{i=1}^{4}{}' \frac{1}{(\alpha_r - \alpha_i)^{\lambda+1}(\alpha_i \beta_i - 1)}, \quad t_{r, \lambda} = \sum_{i=1}^{4}{}' \frac{\beta_r + \beta_i}{(\alpha_r - \alpha_i)^{\lambda+1}(\alpha_i \beta_i - 1)}. \tag{41}$$

Then we find

$$A^{(k)}_{r, j} = \frac{1}{4j(\alpha_r \beta_r - 1)}\left[2 q_r A^{(k-1)}_{r, j-2} - \left(v^2 \frac{\partial A^{(k-1)}_{r, j-1}}{\partial u} + u^2 \frac{\partial A^{(k-1)}_{r, j-1}}{\partial v}\right)\right]$$
$$+ \frac{1}{4} \sum_{\lambda=0}^{2k-j-1} (-1)^\lambda \left\{(-t_{r, \lambda} u + 2 s_{r, \lambda} v) \frac{\partial^\lambda A^{(k-1)}_{r, j+\lambda-1}}{\partial u^\lambda} - \lambda t_{r, \lambda} \frac{\partial^{\lambda-1} A^{(k-1)}_{r, j+\lambda-1}}{\partial u^{\lambda-1}}\right.$$
$$\left. - s_{r, \lambda} u \frac{\partial^{\lambda+1} A^{(k-1)}_{r, j+\lambda}}{\partial u^\lambda \partial v} - \lambda s_{r, \lambda} \frac{\partial^\lambda A^{(k-1)}_{r, j+\lambda}}{\partial u^{\lambda-1} \partial v}\right\}, \tag{42}$$
$$(j = 1, 2, \dots, 2k-1),$$

and a second equivalent form corresponding to the second form of equation (35).

Thus we may obtain a solution of (S) in the form

$$y_r = y_r^0 + y_r^{(1)} c_0 + y_r^{(2)} c_0^2 + \dots\dots + y_r^{(k)} c_0^k + \mathfrak{R},$$

where the further development of $\mathfrak{R}$ would give powers of higher than the k-th order in c_0, k being any finite integer, and each of the functions $y_r^{(i)}$ being of the form

$$y_r^{(i)} = e^{\alpha_r u + \beta_r v} \sum_{j=1}^{2i} A_{r,j}^{(i)}.$$

We may not, however, conclude in this fashion that a solution of (S) exists of the form

$$y_r^{(0)} + y_r^{(1)} c_0 + y_r^{(2)} c_0^2 + \dots\dots \textit{ ad infinitum},$$

since we can not be certain that the limits of the integrals, as determined by equations (39), do not tend to become infinite with k. If this were the case, the convergence theorem which was proved in § 3 would not be applicable.

In order to establish the existence of such solutions, we may proceed as follows: In equation (25) let us put $a = b = 0$. We then find, making use of (30) and (30a),

$$y_r^{(1)} = e^{\alpha_r u + \beta_r v} P_r^{(1)}(u,v) + \sum_{s=1}^{4} c_{r,s}^{(1)} e^{\alpha_s u + \beta_s v},$$

where

$$c_{r,s}^{(1)} = \frac{1}{2(\alpha_s \beta_s - 1)(\alpha_r - \alpha_s)(\beta_r - \beta_s)}, \quad (s \neq r), \quad \sum_{s=1}^{4} c_{r,s}^{(1)} = 0.$$

With this value of $y_r^{(1)}$ we find further

$$y_r^{(2)} = e^{\alpha_r u + \beta_r v} P_r^{(2)}(u,v) + \sum_{s=1}^{4} c_{r,s}^{(1)} \left\{ e^{\alpha_s u + \beta_s v} P_s^{(1)}(u,v) + \sum_{t=1}^{4} c_{s,t}^{(1)} e^{\alpha_t u + \beta_t v} \right\} + \sum_{s=1}^{4} c_{r,s}^{(1,1)} e^{\alpha_s u + \beta_s v},$$

where

$$c_{r,s}^{(1,1)} = -R_{r,i,0}^{(2)}, \quad (s \neq r), \quad \sum_{s=1}^{4} c_{r,s}^{(1,1)} = 0,$$

or

$$y_r^{(2)} = e^{\alpha_r u + \beta_r v} P_r^{(2)}(u,v) + \sum_{s=1}^{4} c_{r,s}^{(1)} e^{\alpha_s u + \beta_s v} P_s^{(1)}(u,v) + \sum_{s=1}^{4} c_{r,s}^{(2)} e^{\alpha_s u + \beta_s v},$$

if we write

$$c_{r,s}^{(2)} = c_{r,s}^{(1,1)} + \sum_{j=1}^{4} c_{r,j}^{(1)} c_{j,s}^{(1)}.$$

We may now prove by induction that

$$y_r^{(k)} = e^{\alpha_r u + \beta_r v} P_r^{(k)}(u,v) + \sum_{s=1}^{4} c_{r,s}^{(1)} e^{\alpha_s u + \beta_s v} P_s^{(k-1)}(u,v) + \dots\dots + \sum_{s=1}^{4} c_{r,s}^{(k)} e^{\alpha_s u + \beta_s v},$$

or

$$y_r^{(k)} = e^{\alpha_r u + \beta_r v} P_r^{(k)}(u,v) + \sum_{s=1}^{4} \sum_{j=1}^{k} c_{r,s}^{(j)} e^{\alpha_s u + \beta_s v} P_s^{(k-j)}(u,v), \tag{36}$$

all of the quantities $c_{r,s}^{(j)}$ being constants, and the functions $P_\mu^{(\lambda)}$ being polynomials of degree 2λ.

We know that the series

$$y_r = e^{\alpha_r u + \beta_r v} + \sum_{k=1}^{\infty} y_r^{(k)} c_0^k \tag{37}$$

is absolutely and uniformly convergent for all finite values of u, v and c_0, and that for $u = v = 0$ we shall have

$$y_r = 1, \quad \frac{\partial y_r}{\partial u} = \alpha_r, \quad \frac{\partial y_r}{\partial v} = \beta_r, \quad \frac{\partial^2 y_r}{\partial u \partial v} = \alpha_r \beta_r. \tag{38}$$

From the general theory of linear partial differential equations, we know further that the uniquely determined solution of system (S), which satisfies the initial conditions (38), may be represented as a power-series $\mathfrak{P}(u, v, c_0)$ in the three variables u, v, c_0, absolutely and uniformly convergent for all finite values of these variables. Moreover, for any system of finite values of u, v, c_0, there exists a convergent series D of positive constants which dominates the series of absolute values of the terms of $\mathfrak{P}(u, v, c_0)$.

Since the series (37) and $\mathfrak{P}(u, v, c_0)$ are both absolutely and uniformly convergent, and since both of these series represent, for all finite values of u, v, c_0, the same solution of (S), we shall obtain $\mathfrak{P}(u, v, c_0)$ from (37), if we substitute in (37) the expressions (36) for $y_r^{(k)}$, expand the exponentials into power-series and perform the indicated multiplications.

Now the quantities $|e^{\alpha_s u + \beta_s v}|$ and $|P_r^{(\lambda)}(u, v)|$ are at most equal to the sum of the moduli of the terms of their respective expansions. Let us replace the several terms $y_r^{(k)} c_0^k$, of series (37), by the sums resulting from (36), and let us regard the individual terms of the sums thus obtained as terms of a new series Σ. The series of the absolute values of the terms of Σ will be dominated by a series of positive constants, whose sum is at most equal to the sum of the series D of positive constants, which dominates the series of absolute values of the terms of $\mathfrak{P}(u, v, c_0)$. Consequently, the series Σ itself and any series composed of terms selected from those of Σ, provided that no term of Σ be taken more than once, will be absolutely and uniformly convergent.

Therefore, the series

$$y = e^{\alpha_r u + \beta_r v} \sum_{k=0}^{\infty} P_r^{(k)}(u, v)\, c_0^k,$$

which is known to satisfy system (S) formally, is absolutely and uniformly convergent. If now we differentiate this series term by term, either once or twice, with respect to u or v, and compare the resulting series with the corresponding first- or second-order derivatives of $\mathfrak{P}(u, v, c_0)$, we recognize by a repetition of the above argument that these series also are absolutely and uniformly convergent and therefore represent $\frac{\partial y}{\partial u}, \frac{\partial y}{\partial v}$, etc.

We finally obtain the following theorem:

If u_0, v_0 *is a point for which the exponents* $\alpha_1, \ldots, \alpha_4$ *are distinct, there exist four linearly independent solutions of system* (S), *of the form*

$$y_r = e^{\alpha_r (u-u_0)+\beta_r (v-v_0)} [1 + \sum_{k=1}^{\infty} P_r^{(k)} (u-u_0, v-v_0)\, c_0^k], \quad (r = 1, 2, 3, 4),$$

where $P_r^{(k)}$, *the coefficient of* c_0^k, *is a polynomial of degree* $2k$.

In formulating this theorem we have made use of a terminology suggested by our discussion of the effect of the linear transformation (28a). In the vicinity of the point u_0, v_0, the coefficients of y in the two equations of (S) may be written in the form

$$c_0 (u - u_0) + c_0 u_0 + c_1, \quad c_0 (v - v_0) + c_0 v_0 + c_2,$$

and we may speak of the four pairs of numbers α, β which satisfy the equations

$$\alpha^2 + 2\beta + c_0 u_0 + c_1 = 0, \quad \beta^2 + 2\alpha + c_0 v_0 + c_2 = 0,$$

as the *exponents for the point* u_0, v_0.

We see that, associated with a system of form (S), there is a set of polynomials. We have seen, moreover, how these polynomials may be calculated by means of certain recursion formulæ. These formulæ, however, are rather complicated, and we now proceed to study the polynomials $P_r^{(k)}(u, v)$ by a different method, which gives far simpler results.

Let us substitute, in (S),

$$y = \eta\, e^{\alpha u + \beta v},$$

where α and β are any two numbers which satisfy the *characteristic* equations

$$\alpha^2 + 2\beta + c_1 = 0, \quad \beta^2 + 2\alpha + c_2 = 0.$$

The system (S) will be transformed into the following system:

$$\eta_{uu} + 2\alpha\eta_u + 2\eta_v + c_0 u\eta = 0, \quad \eta_{vv} + 2\eta_u + 2\beta\eta_v + c_0 v\eta = 0. \tag{40}$$

Since (S) has a solution of the form

$$e^{\alpha u + \beta v} [1 + \sum_{k=1}^{\infty} P^{(k)} (u, v)\, c_0^k],$$

(40) must have a solution of the form

$$\eta = 1 + \sum_{k=1}^{\infty} P^{(k)} (u, v)\, c_0^k, \tag{41}$$

where

$$P^{(k)} (u, v) = \sum_{i=1}^{2k} A_i^{(k)} (u, v), \tag{42}$$

$A_i^{(k)}$ being a homogeneous polynomial of degree i.

If we substitute the expression (41) for η into (40), we find

$$\left.\begin{aligned} P_{uu}^{(k)} + 2\alpha P_u^{(k)} + 2P_v^{(k)} + uP^{(k-1)} = 0, \\ P_{vv}^{(k)} + 2P_u^{(k)} + 2\beta P_v^{(k)} + vP^{(k-1)} = 0, \end{aligned}\right\} \quad (k = 1, 2, 3, \ldots\ldots). \tag{43}$$

From (43) and (42) we find, further,

$$\left.\begin{aligned} \sum_{i=1}^{2k}\left(\frac{\partial^2 A_i^{(k)}}{\partial u^2} + 2\alpha\frac{\partial A_i^{(k)}}{\partial u} + 2\frac{\partial A_i^{(k)}}{\partial v}\right) + u\sum_{i=1}^{2k-2} A_i^{(k-1)} = 0, \\ \sum_{i=1}^{2k}\left(\frac{\partial^2 A_i^{(k)}}{\partial v^2} + 2\frac{\partial A_i^{(k)}}{\partial u} + 2\beta\frac{\partial A_i^{(k)}}{\partial v}\right) + v\sum_{i=1}^{2k-2} A_i^{(k-1)} = 0. \end{aligned}\right\} \tag{44}$$

If we equate to zero the terms of degree $2k-1$, we obtain the equations

$$2\alpha\frac{\partial A_{2k}^{(k)}}{\partial u} + 2\frac{\partial A_{2k}^{(k)}}{\partial v} + uA_{2(k-1)}^{(k-1)} = 0, \quad 2\frac{\partial A_{2k}^{(k)}}{\partial u} + 2\beta\frac{\partial A_{2k}^{(k)}}{\partial v} + vA_{2(k-1)}^{(k-1)} = 0,$$

whence

$$2(\alpha\beta - 1)\frac{\partial A_{2k}^{(k)}}{\partial u} = \tfrac{1}{2} q_u A_{2(k-1)}^{(k-1)}, \quad 2(\alpha\beta - 1)\frac{\partial A_{2k}^{(k)}}{\partial v} = \tfrac{1}{2} q_v A_{2(k-1)}^{(k-1)},$$

where

$$q = -\beta u^2 + 2uv - \alpha v^2, \quad q_u = \frac{\partial q}{\partial u}, \quad q_v = \frac{\partial q}{\partial v}, \tag{45}$$

and where $\alpha\beta - 1$ is different from zero if, as we may assume, α is not a repeated root of the characteristic equations. These equations show that the Jacobian of $A_{2k}^{(k)}$ and q is equal to zero, so that $A_{2k}^{(k)}$ must be a function of q. We easily find

$$A_{2k}^{(k)} = \frac{q^k}{4^k(\alpha\beta - 1)^k k!}, \quad (k = 1, 2, 3, \ldots\ldots), \tag{40}$$

a result which we had already obtained by our previous method.

Since $A_0^{(k)} = 0$, the terms of degree zero and unity, in (44), require special attention. We find

$$\frac{\partial^2 A_2^{(k)}}{\partial u^2} + 2\alpha\frac{\partial A_1^{(k)}}{\partial u} + 2\frac{\partial A_1^{(k)}}{\partial v} = 0, \quad \frac{\partial^2 A_2^{(k)}}{\partial v^2} + 2\frac{\partial A_1^{(k)}}{\partial u} + 2\beta\frac{\partial A_1^{(k)}}{\partial v} = 0,$$

whence

$$\begin{aligned} 2(\alpha\beta - 1)\frac{\partial A_1^{(k)}}{\partial u} &= -\beta\frac{\partial^2 A_2^{(k)}}{\partial u^2} + \frac{\partial^2 A_2^{(k)}}{\partial v^2}, \\ 2(\alpha\beta - 1)\frac{\partial A_1^{(k)}}{\partial v} &= +\frac{\partial^2 A_2^{(k)}}{\partial u^2} - \alpha\frac{\partial^2 A_2^{(k)}}{\partial v^2}. \end{aligned}$$

Let us multiply the members of these equations by u and v respectively, and apply Euler's theorem on homogeneous functions. We find

$$A_1^{(k)} = \frac{1}{2(\alpha\beta - 1)}\,\Omega(A_2^{(k)}), \tag{46}$$

where Ω denotes the operator

$$\Omega = (-\beta u + v)\frac{\partial^2}{\partial u^2} + (u - \alpha v)\frac{\partial^2}{\partial v^2} = \frac{1}{2}\left(q_u \frac{\partial^2}{\partial u^2} + q_v \frac{\partial^2}{\partial v^2}\right), \tag{47}$$

which plays a fundamental rôle in this theory. In the same way we find, from (44), equating to zero the terms of the first degree,

$$A_2^{(k)} = \frac{1}{4(\alpha\beta - 1)}\,\Omega\,(A_3^{(k)}). \tag{48}$$

Let us now equate to zero the terms of degree $2k - i - 1$ in (44), where $1 < 2k - i - 1 < 2k - 1$. A precisely similar treatment gives the following equation:

$$2(2k - i)(\alpha\beta - 1)\,A_{2k-i}^{(k)} = q\,A_{2(k-1)-i}^{(k-1)} + \Omega\,(A_{2k-i+1}^{(k)}).$$

If we put $2k - i = \nu$, we find

$$A_\nu^{(k)} = \frac{\Omega\,(A_{\nu+1}^{(k)}) + q\,A_{\nu-2}^{(k-1)}}{2\nu(\alpha\beta - 1)}, \quad (\nu = 1, 2, 3, \ldots, 2k - 1), \tag{49}$$

for this formula will also be valid for $\nu = 1$ or 2 if we equate to zero $A_i^{(k)}$ whenever i becomes zero or negative.

By means of formulæ (40) and (49) all of the homogeneous polynomials $A_i^{(k)}$ may be calculated. We shall actually write down their values for $k = 1$ and $k = 2$, viz.:

$$\left.\begin{aligned} A_1^{(1)} &= \frac{1}{4(\alpha\beta - 1)^2}\left[(\beta^2 - \alpha)u + (\alpha^2 - \beta)v\right], \\ A_1^{(2)} &= \frac{1}{4(\alpha\beta - 1)}\left[-\beta u^2 + 2uv - \alpha v^2\right] = \frac{q}{4(\alpha\beta - 1)}, \end{aligned}\right\} \tag{50}$$

and

$$\left.\begin{aligned} &A_1^{(2)} = \\ &\frac{(-6\alpha+11\beta^2-9\alpha^2\beta+5\alpha^4+4\alpha\beta^3-5\beta^5)u+(-6\beta+11\alpha^2-9\alpha\beta^2+5\beta^4+4\alpha^3\beta-5\alpha^5)v}{32(\alpha\beta-1)^5}, \\ &A_2^{(2)} = \\ &\frac{(-6\beta+5\alpha^2-4\alpha\beta^2+5\beta^4)u^2+2(2+7\alpha\beta-5\alpha^3-5\beta^3+\alpha^2\beta^2)uv+(-6\alpha+5\beta^2-4\alpha^2\beta+5\alpha^4)v^2}{32(\alpha\beta-1)^4}, \\ &A_3^{(2)} = \\ &\frac{(2+3\alpha\beta-5\beta^3)u^3+3(-4\alpha+5\beta^2-\alpha^2\beta)u^2v+3(-4\beta+5\alpha^2-\alpha\beta^2)uv^2+(2+3\alpha\beta-5\alpha^3)v^3}{48(\alpha\beta-1)^3}, \\ &A_4^{(2)} = \\ &\frac{\beta^2u^4-4\beta u^3v+2(\alpha\beta+2)u^2v^2-4\alpha uv^3+\alpha^2v^4}{32(\alpha\beta-1)^2} = \frac{q^2}{32(\alpha\beta-1)^2}. \end{aligned}\right\} \tag{51}$$

We observe in all of these cases that

$$A_\nu^{(k)} = \frac{B_\nu^{(k)}}{(\alpha\beta - 1)^{3k-\nu}}, \tag{52}$$

where $B_\nu^{(k)}$ is a homogeneous polynomial of degree ν in u and v, whose coefficients are integral rational functions of α, β of degree $3k-\nu$. Making use of equation (49), it is easy to prove by induction that this law is true in general.

If we make the substitution (52), our recursion formulæ become

$$B_{2k}^{(k)} = \frac{q^k}{4^k k!}, \quad B_\nu^{(k)} = \frac{1}{2\nu}\left[\Omega\left(B_{\nu+1}^{(k)}\right) + q B_{\nu-2}^{(k-1)}\right], \quad (\nu = 1, 2, \ldots, 2k-1), \tag{53}$$

where we must remember that $B_0^{(j)} = B_{-1}^{(j)} = 0$.

By repeated application of these formulæ we find

$$B_\nu^{(k)} = \frac{\Omega^{2k-\nu}(q^k)}{2^{4k-\nu} k!\, \nu(\nu+1) \ldots (2k-1)} + \sum_{j=1}^{2k-\nu} \frac{\Omega^{2k-\nu-j}\left(q B_{2(k-1)-j}^{(k-1)}\right)}{2^{2k-\nu-j+1} \nu(\nu+1) \ldots (2k-j)},$$
$$(\nu = 1, 2, \ldots, 2k-1), \tag{54}$$

where the symbol Ω^λ indicates that the operation Ω is to be performed λ times in succession.

We may obtain another formula for $B_\nu^{(k)}$ from the equations

$$B_\nu^{(k)} = \frac{1}{2\nu}\left[q B_{\nu-2}^{(k-1)} + \Omega\left(B_{\nu+1}^{(k)}\right)\right],$$
$$B_{\nu-2}^{(k)} = \frac{1}{2(\nu-2)}\left[q B_{\nu-4}^{(k-2)} + \Omega\left(B_{\nu-1}^{(k-1)}\right)\right],$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

by eliminating the first terms occurring in the right members. In this process we must distinguish two cases, according as ν is even or odd. We find

$$\left.\begin{aligned} B_{2\lambda}^{(k)} &= \sum_{i=0}^{\lambda-1} \frac{q^i \Omega\left(B_{2(\lambda-i)+1}^{(k-i)}\right)}{4^{i+1} \lambda(\lambda-1) \ldots (\lambda-i)}, \\ B_{2\lambda-1}^{(k)} &= \sum_{i=0}^{\lambda-1} \frac{q^i \Omega\left(B_{2(\lambda-i)}^{(k-i)}\right)}{2^{i+1} (2\lambda-1)(2\lambda-3) \ldots (2\lambda-2i-1)}, \end{aligned}\right\} \tag{55}$$

or, to include both cases,

$$B_\nu^{(k)} = \sum_{i=0}^{\lambda-1} \frac{q^i \Omega\left(B_{\nu-2i+1}^{(k-i)}\right)}{2^{i+1} \nu(\nu-2) \ldots (\nu-2i)}, \tag{56}$$

where $\lambda = \frac{\nu}{2}$ if ν is even, and $\lambda = \frac{\nu+1}{2}$ if ν is odd.

In the second equation of (55), let us put $\lambda = k$. We find

$$B_{2k-1}^{(k)} = \sum_{i=0}^{k-1} \frac{q^i \Omega\left(q^{k-i}\right)}{2^{2k-i+1} (2k-1)(2k-3) \ldots (2k-2i-1)(k-i)!}.$$

Moreover,

$$\Omega\left(q^{k-i}\right) = 2(k-i)\, q^{k-i-1} l_1 + 4(k-i)(k-i-1)\, q^{k-i-2} l_3, \tag{57}$$

where

$$\left.\begin{aligned} l_1 &= (\beta^2-\alpha)\,u + (\alpha^2-\beta)\,v, \\ l_3 &= (1-\beta^3)\,u^3 + 3\,(\beta^2-\alpha)\,u^2 v + 3\,(\alpha^2-\beta)\,u v^2 + (1-\alpha^3)\,v^3. \end{aligned}\right\} \qquad (58)$$

Consequently,

$$B^{(k)}_{2k-1} = \frac{q^{k-2}}{4^k}\,[q\,l_1\,\theta_1(k) + l_3\,\theta_2(k)], \qquad (59)$$

where

$$\left.\begin{aligned} \theta_1(k) &= \sum_{i=0}^{k-1} \frac{2^i}{(2k-1)(2k-3)\ldots.(2k-2i-1)(k-i-1)!}, \\ \theta_2(k) &= \sum_{i=0}^{k-1} \frac{2^{i+1}}{(2k-1)(2k-3)\ldots.(2k-2i-1)(k-i-2)!}. \end{aligned}\right\} \qquad (60)$$

We may evaluate these sums and thereby very materially simplify equation (59) by the following method: If we equate to zero the terms of degree $2k-2$ in (44), a simple combination of the two resulting equations gives

$$q_v\frac{\partial B^{(k)}_{2k-1}}{\partial u} - q_u\frac{\partial B^{(k)}_{2k-1}}{\partial v} = (\alpha\beta-1)\left(v\frac{\partial^2 B^{(k)}_{2k}}{\partial u^2} - u\frac{\partial^2 B^{(k)}_{2k}}{\partial v^2}\right). \qquad (61)$$

In this equation let us substitute the values (53) and (59) for $B^{(k)}_{2k}$ and $B^{(k)}_{2k-1}$. The left member of (61) becomes

$$L = \frac{\theta_1(k)}{4^k}\,q^{k-1}\,[(\beta^2-\alpha)\,q_v - (\alpha^2-\beta)\,q_u] + \frac{\theta_2(k)}{4^k}\,q^{k-2}\left(q_v\frac{\partial l_3}{\partial u} - q_u\frac{\partial l_3}{\partial v}\right),$$

and the right member reduces to

$$R = \frac{\alpha\beta-1}{4^k\,(k-1)\,!}\,[q^{k-1}\,(v\,q_{uu} - u\,q_{vv}) + (k-1)\,q^{k-2}\,(v\,q_u^2 - u\,q_v^2)]\,.$$

But

$$\begin{aligned} &(\beta^2-\alpha)\,q_v - (\alpha^2-\beta)\,q_u = 2\,(\alpha\beta-1)\,(\alpha u-\beta v), \\ &\qquad q_v\frac{\partial l_3}{\partial u} - q_u\frac{\partial l_3}{\partial v} = 6\,(\alpha\beta-1)\,[-u^3 + (2\alpha+\beta^2)\,u^2 v - (2\beta+\alpha^2)\,u v^2 + v^3], \\ &\qquad v\,q_{uu} - u\,q_{vv} = 2\,(\alpha u - \beta v), \\ &\qquad v\,q_u^2 - u\,q_v^2 = 4\,[-u^3 + (2\alpha+\beta^2)\,u^2 v - (2\beta+\alpha^2)\,u v^2 + v^3], \end{aligned}$$

so that the equation $L = R$ assumes the form

$$\begin{aligned} &q\,(\alpha u-\beta v)\left[\theta_1(k) - \frac{1}{(k-1)\,!}\right] \\ &\qquad + [-u^3 + (2\alpha+\beta^2)\,u^2 v - (2\beta+\alpha^2)\,u v^2 + v^3]\left[3\,\theta_2(k) - \frac{2}{(k-2)\,!}\right] = 0. \end{aligned}$$

The two cubic expressions which appear in the left member of this equation are linearly independent, since $\alpha\beta - 1$ is different from zero. Consequently we must have

$$\theta_1(k) = \frac{1}{(k-1)!}, \quad \theta_2(k) = \frac{2}{3(k-2)!}, \tag{62}$$

so that

$$B_{2k-1}^{(k)} = \frac{q^{k-2}\,[3\,q\,l_1 + 2\,(k-1)\,l_3]}{3\cdot 4^k\,(k-1)!}. \tag{63}$$

The explicit expression for $B_{2k-2}^{(k)}$, obtained by an extension of this method, is already rather complicated, viz.:

$$\begin{aligned} B_{2k-2}^{(k)} = \frac{2\,q^{k-4}}{3^2\cdot 4^{k+1}\,(k-2)!}\,[18\{2(k-2)-3(\alpha\beta-1)\}\,q^3 \\ +9\{5l_1^2+4(k-2)(\alpha\beta-1)uv-8(\alpha\beta-1)^2uv\}\,q^2 \\ +6\{(k-2)(2l_1l_3-3l_4)+3(k-2)(\alpha\beta-1)^2u^2v^2\}\,q \\ +4(k-2)(k-3)\,l_3^2], \end{aligned} \tag{64}$$

where

$$l_4 = \beta\,(-\beta u + v)^4 + \alpha\,(u - \alpha v)^4. \tag{65}$$

The transformation (28a) enables us to simplify these formulæ very considerably. Let us assume that such a transformation has been made so as to reduce c_1 and c_2 to zero. The characteristic equations become

$$\alpha^2 + 2\,\beta = \beta^2 + 2\,\alpha = 0,$$

and one pair of solutions is $\alpha = \beta = 0$. The operator Ω reduces to

$$\Omega = v\,\frac{\partial^2}{\partial u^2} + u\,\frac{\partial^2}{\partial v^2},$$

and it becomes an easy matter to calculate as many of the polynomials $P^{(k)}(u, v)$ as may be desired, although, even with this simplification, the general expression seems to be rather complicated. We find

$$\left.\begin{aligned} P^{(0)} &= 1, \\ P^{(1)} &= -\frac{1}{2}\,u\,v, \\ P^{(2)} &= -\frac{1}{8}\,u\,v - \frac{1}{24}\,(u^3+v^3) + \frac{1}{8}\,u^2v^2, \\ P^{(3)} &= -\frac{3}{32}\,u\,v + \frac{1}{32}\,(u^3+v^3) - \frac{3}{32}\,u^2v^2 + \frac{1}{48}\,u\,v\,(u^3+v^3) - \frac{1}{48}\,u^3v^3, \\ P^{(4)} &= \frac{7}{2^6}\,u\,v - \frac{7}{2^6\cdot 3}\,(u^3+v^3) + \frac{7}{2^6}\,u^2v^2 - \frac{11}{2^7\cdot 3}\,u\,v\,(u^3+v^3) \\ &\quad + \frac{1}{2^7\cdot 3^2}\,[(u^3+v^3)^2+36\,u^3v^3] - \frac{1}{2^6\cdot 3}\,u^2v^2(u^3+v^3) + \frac{u^4v^4}{2^4\cdot 4!}. \end{aligned}\right\} \tag{66}$$

We shall close this discussion with one further remark. The expression (41) for η may be written

$$\eta = e^{\frac{c_0 q}{4(\alpha\beta-1)}} + \sum_{k=1}^{\infty} Q^{(k)}(u, v)\,c_0^k, \tag{67}$$

where the functions $Q^{(k)}(u, v)$ are polynomials of degree $2k-1$, differing from $P^{(k)}(u, v)$ merely by the absence of the terms of degree $2k$. Thus, by introduction of the first exponential term of (67), the polynomials $P^{(k)}$ have been deprived of their terms of highest degree. By means of equation (63) this process may be carried out one step further, depriving the polynomials $Q^{(k)}$, in their turn, of their terms of highest degree. However, for the present, we shall refrain from any further developments in this direction.

§ 5. *Integral Equations Satisfied by the Solutions of* (*S*).

If Y is any solution of (S), it may also be regarded as a solution of the non-homogeneous system

$$y_{uu}+2y_v+c_1 y+c_0 u Y=0, \qquad y_{vv}+2y_u+c_2 y+c_0 v Y=0.$$

The method of § 2 for integrating such a system is applicable, since the integrability conditions are all satisfied. Consequently we obtain the following relation:

$$(J) \quad Y=c_0\sum_{i=1}^{4} e^{\alpha_i u+\beta_i v}\int_{a,b}^{u,v}\frac{e^{-\alpha_i u-\beta_i v}}{4(\alpha_i\beta_i-1)}\left[\left\{(-\beta_i u+2v)Y-u\frac{\partial Y}{\partial v}\right\}du+\left\{(2u-\alpha_i v)Y-v\frac{\partial Y}{\partial u}\right\}dv\right]+\sum_{i=1}^{4} a_i e^{\alpha_i u+\beta_i v},$$

where $a_1, \ldots, a_4$ are constants. The integrals on the right member of this equation are independent of the path of integration, as a consequence of the differential equations satisfied by Y, and have definite finite values since all of the solutions of (S) are integral transcendental functions of u and v.

Moreover, if these integrals be denoted by l_i, they have been determined, in accordance with the method of § 2, in such a way that, if we write

$$\eta=\sum_{i=1}^{4} l_i e^{\alpha_i u+\beta_i v},$$

we shall have

$$\frac{\partial\eta}{\partial u}=\sum_{i=1}^{4}\alpha_i l_i e^{\alpha_i u+\beta_i v}, \quad \frac{\partial\eta}{\partial v}=\sum_{i=1}^{4}\beta_i l_i e^{\alpha_i u+\beta_i v}, \quad \frac{\partial^2\eta}{\partial u\,\partial v}=\sum_{i=1}^{4}\alpha_i\beta_i l_i e^{\alpha_i u+\beta_i v},$$

just as though $l_1, \ldots, l_4$ were constants. Consequently, the solution of the integral equation (J) will satisfy the same initial conditions for $u=a$, $v=b$ as the function

$$\sum_{i=1}^{4} a_i e^{\alpha_i u+\beta_i v},$$

so that the equation of form (J) satisfied by y_r will be obtained if we replace this sum by $e^{\alpha_r u + \beta_r v}$.

Let us change the notation for the variables of integration. Then, the integral equation satisfied by y_r may be written as follows:

$$(J_r) \quad Y(u, v) = e^{\alpha_r u + \beta_r v}$$
$$+ c_0 \int_{a,b}^{u,v} \sum_{i=1}^{4} \frac{e^{\alpha_i (u-p) + \beta_i (v-q)}}{4(\alpha_i \beta_i - 1)} \left[\left\{ (-\beta_i p + 2q) Y(p,q) - p \frac{\partial Y(p,q)}{\partial q} \right\} dp \right.$$
$$\left. + \left\{ (2p - \alpha_i q) Y(p,q) - q \frac{\partial Y(p,q)}{\partial p} \right\} dq \right].$$

Evidently the Liouville-Neumann method of successive substitution, as applied to this integral equation, will lead us back to the solution already obtained in § 3.

It is easy to see, conversely, that if α_i, β_i are the four pairs of solutions (supposed distinct) of the equations

$$\alpha^2 + 2\beta + c_1 = 0, \qquad \beta^2 + 2\alpha + c_2 = 0,$$

any continuous solution of (J_r), with continuous first and second derivatives, will be a solution of system (S), satisfying the given initial conditions for $u = a$, $v = b$, thus establishing, by a second method, the uniqueness of the corresponding solution of the integral equation (J_r).

In fact, we find from (J_r) by direct differentiation

$$Y_{uu} + 2Y_v + c_1 Y = c_0 \sum_{i=1}^{4} \frac{1}{4(\alpha_i \beta_i - 1)} \left[(-\beta_i + 4u - \alpha_i \beta_i u) Y + \beta_i u \frac{\partial Y}{\partial u} \right.$$
$$\left. - (1 + \alpha_i u) \frac{\partial Y}{\partial v} - u \frac{\partial^2 Y}{\partial u \partial v} \right]. \qquad (68)$$

But we have from (24)

$$\frac{1}{\alpha_i \beta_i - 1} = \frac{2k_i}{\Delta},$$

and $\frac{1}{2} k_i$ is the co-factor of $\alpha_i \beta_i$ in the determinant Δ defined by (22). Consequently,

$$\sum_{i=1}^{4} \frac{1}{\alpha_i \beta_i - 1} = \sum_{i=1}^{4} \frac{\alpha_i}{\alpha_i \beta_i - 1} = \sum_{i=1}^{4} \frac{\beta_i}{\alpha_i \beta_i - 1} = 0, \quad \sum_{i=1}^{4} \frac{\alpha_i \beta_i}{\alpha_i \beta_i - 1} = 4,$$

so that the right member of (68) reduces to $-c_0 u Y$. In the same way we may prove that Y must also satisfy the second equation of system (S).

The integral equations (J) and (J_r) are simple examples of a type of integral equations which apparently has not yet been considered. The characteristic property of equations of this type is that they contain open line integrals subject to the condition of being independent of the path. We shall leave the general discussion of integral equations of this kind for a future occasion.

We shall close this section by pointing out a remarkable formula for the partial derivative, with respect to c_0, of any solution of system (S). Let

$$\frac{\partial y}{\partial c_0} = \eta.$$

Then we see that η is a solution of the non-homogeneous system

$$(S') \qquad \begin{aligned} &\eta_{uu} + 2\eta_v + (c_0 u + c_1)\eta + uy = 0, \\ &\eta_{vv} + 2\eta_u + (c_0 v + c_1)\eta + vy = 0. \end{aligned}$$

The corresponding homogeneous system is (S). If we apply the method of § 2 to the integration of system (S'), we obtain $\frac{\partial y}{\partial c_0}$ expressed as a sum of integrals of exact differentials depending upon the solutions $y_1, \ldots, y_4$ of (S). Of course, $\frac{\partial y}{\partial c_0}$ may also be regarded as a solution of an integral equation obtained from (J) by differentiation with respect to c_0.

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